

Lagrangian particle statistics in turbulent flows from a simple vortex model

M. Wilczek,^{1,*} F. Jenko,² and R. Friedrich¹

¹*Institute for Theoretical Physics, University of Münster, Wilhelm-Klemm-Strasse 9, D-48149 Münster, Germany*

²*Max-Planck-Institut für Plasmaphysik, Boltzmannstrasse 2, D-85748 Garching, Germany*

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The statistics of Lagrangian particles in turbulent flows is considered in the framework of a simple vortex model. Here, the turbulent velocity field is represented by a temporal sequence of Burgers vortices of different circulation, strain, and orientation. Based on suitable assumptions about the vortices' statistical properties, the statistics of the velocity increments is derived. In particular, the origin and nature of small-scale intermittency in this model is investigated both numerically and analytically. We critically compare our results to experimental studies.

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I. INTRODUCTION

Our understanding of the spatiotemporal properties of turbulent flows is still fragmentary [1–3]. In recent years, however, significant progress could be achieved by focusing on the dynamics of Lagrangian particles (see, e.g., Ref. [4], and references therein). Especially laboratory experiments have given useful insight to important Lagrangian statistical properties. For example, the statistics of Lagrangian velocity increments [5,6] and those of the turbulent acceleration of a tracer particle [7,5] have been obtained experimentally. Moreover, direct numerical simulations on large-scale computers have been performed [8] which allow to study particle trajectories and their statistical properties from yet another point of view. While all results obtained from laboratory and computer experiments seem to be consistent so far, the underlying physical mechanisms are still subject to discussion. In particular, beyond phenomenological approaches such as the multifractal model [9], it has not yet been possible to arrive at a theoretical derivation of the single particle velocity increment distribution for a fully developed turbulent flow from first principles.

There are hints, though, that certain turbulent structures are largely responsible for the statistical properties of Lagrangian particles. It has been shown by direct numerical simulations [10,11] and by thorough experimental investigations [12] that the turbulent field contains elongated vortex filaments which can be interpreted as Burgers vortices. The latter are well-known solutions of the Navier-Stokes equation [13]. In fact, these vortices have been called the sinews of turbulence by Moffatt and co-workers [14]. Hatakeyama and Kambe recently modeled turbulent fields by a random arrangement of such Burgers vortices [15]. Interestingly, through this procedure they were able to reproduce the multifractal scaling behavior of the longitudinal structure functions.

In the present paper, we shall employ a similar approach. We will construct a minimal model that reproduces one of the central Lagrangian observations, namely, the transition from Gaussian velocity increment probability density func-

tions to fat-tail probability density functions (PDFs). We focus on the most simple formulation of the model. However, parameters arising in this model are chosen in accordance with available information gained from experiments and numerical calculations. To be more precise, we will consider the Lagrangian statistics of a tracer particle in a turbulent velocity field which is modeled by a temporal sequence of Burgers vortices of different circulation, strain, and orientation. Such a model is motivated by the fact that Navier-Stokes turbulence is known to create strong vortex filaments which tend to dominate the time evolution of Lagrangian particles. This view is supported, e.g., by the recent work of Biferale and co-workers [8] who detect so-called “vortex trapping events” of tracer particles in their direct numerical simulations. Hence we are led to consider the path of a Lagrangian particle in the field of a single Burgers vortex. After a certain lifetime T_l , this vortex decays and is replaced by another vortex which differs in circulation Γ , strain-parameter a , and orientation. Then, the process starts anew. Making suitable assumptions about the vortices' statistical properties, we will be able to determine the statistics of the velocity increments in the framework of our model this way. In particular, we will be able to investigate the origin and nature of small-scale intermittency (i.e., within the dissipative range) numerically and partly analytically within the present model. For the case of single vortex filament an analytical formula will be derived. Furthermore a numerical implementation of the physically motivated model will be presented.

The remainder of the present article is structured as follows. Starting from the exact solution of a particle trajectory in a Burgers vortex, we will investigate the functional structure of the PDF of the velocity increment in Sec. II. We will present results of a Lagrangian particle evolving through a temporal sequence of trapping events in Sec. III. There, especially the evolution of the PDFs of the velocity increments is considered. In Sec. IV the results are compared to experimental results and discussed.

II. LAGRANGIAN PARTICLE IN A SINGLE BURGERS VORTEX

The velocity field of a single Burgers vortex with circulation Γ in a strain field $\mathbf{u}_{st}(\mathbf{x}, t) = [-\frac{a}{2}x, -\frac{a}{2}y, az]$ is given by

*mwilczek@uni-muenster.de

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_{\text{st}}(\mathbf{x}, t) + \mathbf{e}_z \times \mathbf{e}_r \frac{\Gamma}{2\pi r} [1 - e^{-ar^2/(4\nu)}], \quad (1)$$

where ν denotes the kinematic viscosity of the fluid, Γ denotes the circulation of the vortex, and $r = \sqrt{x^2 + y^2}$. \mathbf{e}_z and \mathbf{e}_r denote the unit vectors in the z and radial directions, respectively. Switching now to the Lagrangian frame, the evolution of a Lagrangian particle starting from an initial position \mathbf{x}_0 is determined by

$$\frac{d\mathbf{X}}{dt}(\mathbf{x}_0, t) = [\mathbf{u}(\mathbf{x}, t)]_{\mathbf{x}=\mathbf{X}(\mathbf{x}_0, t)}, \quad (2)$$

$$\frac{d^2\mathbf{X}}{dt^2}(\mathbf{x}_0, t) = [-\nabla p(\mathbf{x}, t) + \nu\Delta\mathbf{u}(\mathbf{x}, t)]_{\mathbf{x}=\mathbf{X}(\mathbf{x}_0, t)}, \quad (3)$$

which means that for the velocity $\dot{\mathbf{X}}$ of a tracer particle one has to evaluate the velocity field \mathbf{u} at the position of the tracer particle \mathbf{X} . The acceleration acting on the particle then is given by negative gradient of the pressure p and viscous contributions at the position of the tracer particle. Due to the axial symmetry of the underlying Eulerian velocity field, we seek for solutions of the form

$$\mathbf{x}(t) = r(t)\mathbf{e}_r[\varphi(t)] + z(t)\mathbf{e}_z, \quad (4)$$

so that $r(t)$, $\varphi(t)$, and $z(t)$ have to obey the following set of differential equations:

$$\dot{r} = -\frac{a}{2}r, \quad (5)$$

$$\dot{\varphi} = \frac{1}{r}u_\varphi(r), \quad (6)$$

$$\dot{z} = az, \quad (7)$$

with $u_\varphi = \frac{\Gamma}{2\pi r}[1 - e^{-ar^2/(4\nu)}]$. The solutions for two of these components obviously read

$$z(t) = z_0 e^{at}, \quad (8)$$

$$r(t) = r_0 e^{-a/2t}. \quad (9)$$

So the z position of the particle increases exponentially in time. Since we want to focus on the importance of the oscillatory motion around the axis of the vortex, we have to study the case of low straining, i.e., the factor at has to be small compared to the circulation Γ . Plugging the solution for $r(t)$ into the differential equation for the azimuthal component one obtains

$$\dot{\varphi}(t) = \frac{\Gamma}{2\pi r_0^2} e^{at} (1 - e^{-a/(4\nu)r_0^2 e^{-at}}). \quad (10)$$

The solution of this differential equation is thereby reduced to an integration. However this integral cannot be solved explicitly, and hence we want to restrict our following calculations to the limiting cases where the particle is far away from the vortex's axis or near the core. For the farfield case the point-vortex approximation holds, leading to an angular velocity of

$$\dot{\varphi}(t) = \frac{\Gamma}{2\pi r_0^2} e^{at}. \quad (11)$$

The angular velocity of the particle decays such as $\frac{1}{r_0^2}$ leading to a differential rotation. In case of low straining, that is a is nearly zero, the exponential can be approximated by unity in lowest order. In this approximation the solution is

$$\varphi(t) = \frac{\Gamma}{2\pi r_0^2} t = \omega(\Gamma, r_0)t. \quad (12)$$

The second limiting case is where the particle is near the viscous core of the vortex. In this case we can expand the exponential in Eq. (10) into a series. The first-order approximation then is

$$\dot{\varphi}(t) = \frac{\Gamma a}{8\pi\nu}, \quad (13)$$

which leads to

$$\varphi(t) = \frac{\Gamma a}{8\pi\nu} t = \omega(\Gamma, a)t. \quad (14)$$

This means that near the viscous core of the vortex the motion is given by a rigid body rotation determined by the parameters a , Γ , and ν . In order to clarify the functional structure of the following results for the velocity increment distribution we will first calculate the simple case, where only one vortex is involved.

Statistical observations on turbulence often focus on the velocity increments $v_x(\tau)$ of a single Cartesian component with a time lag τ (see, for example, Ref. [6]) and on the turbulent acceleration [16]. For the case of small τ the evolution of a particle in a turbulent flow is dominated by the nearest vortex filament. So the statistics for small τ can be related to the dynamical equations of a single vortex.

Hence we have to write down the discussed solution in Cartesian coordinates. The x component of the position of the Lagrangian particle is given by

$$x(t) = r_0 e^{-(a/2)t} \cos \varphi(t). \quad (15)$$

Intermittency in the velocity signal is often said to be caused by the strong accelerations of a vortex filament [17]. These accelerations originate from the oscillation of the particle round the axis of the vortex [18]. We now neglect the radial transport of the particle in order to focus on these rotational accelerations. This approximation holds whenever the product at is small compared to the vorticity determined by Γ . As we want to focus on the impact of strong vortex filaments on the statistics this is a reasonable approximation. Neglecting the straining the x component of the velocity reads

$$u_x(t) = -r_0 \omega \sin \omega t, \quad (16)$$

with ω being the oscillation frequency given by the two limiting cases discussed above. This leads to a simple equation for the velocity increments

$$v_x(\tau) = u_x(t + \tau) - u_x(t) = -r_0\omega[\sin\omega(t + \tau) - \sin\omega t]. \quad (17)$$

Some simple trigonometric relations turn this expression to

$$v_x(\tau) = -2r_0\omega \left| \sin \frac{\omega\tau}{2} \right| \sin(\omega t + \psi) \equiv A(\tau)\sin(\omega t + \psi), \quad (18)$$

with a phase ψ which depends on the initial conditions. Note that the velocity increments oscillate similar to the velocity components themselves, being modulated by an amplitude which depends on the parameters of the vortex as well as on the time lag τ . The next aim is to deduce the corresponding probability density function. Instead of taking a time average, we average over the phase $\alpha = \omega t + \psi$, which can be assumed to be uniformly distributed

$$\begin{aligned} f(v_x) &= \frac{1}{2\pi} \int_0^{2\pi} d\alpha \delta(v_x - A \sin\alpha) \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\alpha \frac{1}{|A \cos\alpha|} \delta\left(\alpha - \arcsin \frac{v_x}{A}\right). \end{aligned} \quad (19)$$

Noting that the violation of $v_x < A$ leads to imaginary values of the probability, the solution reads

$$f(v_x) = \text{Re} \left(\frac{1}{\pi\sqrt{A^2 - v_x^2}} \right) = \text{Re} \left(\frac{1}{\pi\sqrt{4r_0^2\omega^2 \sin^2\left(\frac{\omega\tau}{2}\right) - v_x^2}} \right). \quad (20)$$

The dependence on the parameters Γ and a is absorbed into the frequency ω . In order to obtain our result we have to average over the vorticity parameter Γ , the strain parameter a and the initial position of the particle r_0 , respectively. Let $f(\Gamma)$, $g(a)$, and $h(r_0)$ be the corresponding distributions of the parameters. (This notation implies statistical independence, a restriction, which does not have to be made.) The probability distribution of the velocity increments is then given by

$$p(v_x) = \text{Re} \left(\int \frac{f(\Gamma)g(a)h(r_0)d\Gamma da r_0 dr_0}{\pi\sqrt{4r_0^2\omega^2 \sin^2\left(\frac{\omega\tau}{2}\right) - v_x^2}} \right). \quad (21)$$

This expression explicitly reveals the dependence of the probability density functions on the distribution of the vorticity parameter Γ . The structure of the functional form of the PDF shows that the shape of the PDF is given by a superposition of functions of the form $1/\sqrt{A^2 - x^2}$ weighted by factors from the physical parameter distributions. The complex interplay of the parameter distribution then leads to the fat-tail PDF's, as observed in the sections presented below.

III. LAGRANGIAN PARTICLE IN A SEQUENCE OF VORTICES

As discussed above, we model the evolution of a Lagrangian particle in a turbulent flow by a temporal sequence of randomly oriented coherent structures. In particular, a particle is exposed to the velocity field of a single vortex for a given lifetime T_l . In our model T_l is given by the typical lifetime of an eddy in the dissipative range. According to Ref. [8], this lifetime is of the order of $10\tau_\eta$ [$\tau_\eta = (\epsilon/\nu)^{1/2}$ denotes the Kolmogorov time scale determined by the energy dissipation and the kinematic viscosity]. In real turbulent flows, viscosity, or vortex merging processes then force this vortex to decay. This is crudely modeled by simply turning the vortex off. Subsequently another vortex is switched on at a certain distance. This distance is given by the typical spatial density of strong vortex filaments in a turbulent flow (see, for example, Ref. [3], Chap. 7.3). The orientation of the vortex's axis in the three-dimensional space is chosen randomly. As a result the statistics become isotropic.

In order to render the model physically sound, some properties of turbulent flows have to be taken into account. In the following we will express all dependent parameters in terms of the energy dissipation ϵ and the kinematic viscosity ν , for which we choose typical values motivated by experiments (see, for example, Ref. [12]). In addition special interest has to be put on the statistical properties of the strain-parameter a and the circulation Γ . A hint on the distribution of Γ is given by the authors of Ref. [10], who measured the distribution of the strengths of the vortices directly from numerical simulations. In our model this is taken into account by

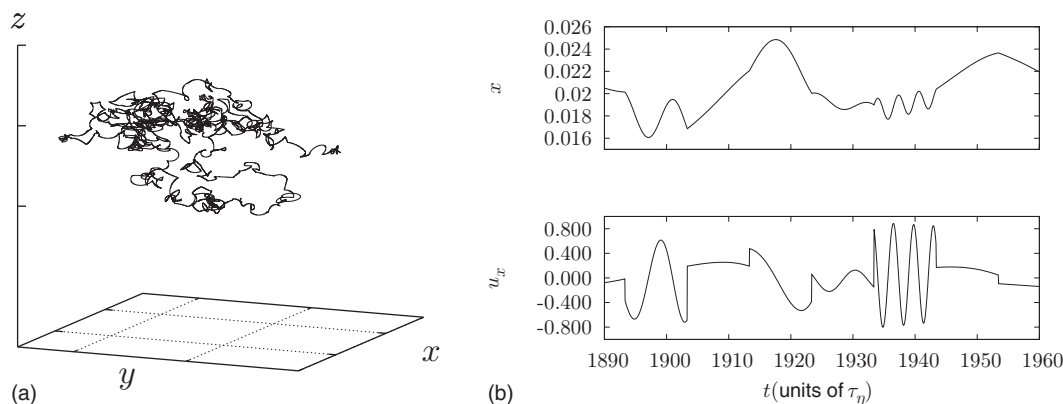


FIG. 1. Trajectory of a Lagrangian particle. t is given in multiples of τ_η .

modeling the distribution of Γ with a log-normal distribution. By this choice we have extreme-valued vortex events with a nonvanishing probability. However, also low-valued vortices are modeled frequently. This accounts to the fact that a Lagrangian particle does not encounter one intense vortex after another, but also runs through periods of low vorticity.

Also the strain parameter a is chosen randomly. This means, that in our model vortices of different radii appear, as the strain parameter a directly affects the radius of the Burgers vortex. However, this distribution cannot be equally distributed as we want to model small-scale vortices. Therefore a peaked distribution is needed. The expectation value of this distribution is related to the typical size of the smallest vortex filaments. We choose this distribution to be Gaussian with standard deviation $\sigma_a=1$. This choice is somewhat arbitrary, but it was ensured that the numerical results vary only little over a wide range of possible values of σ_a . The expectation value of the distribution can be derived from the definition of the radius of the Burgers vortex. This typical scale is defined as $r_B=(\frac{4\nu}{a})^{1/2}$. If we demand r_B to be of the order of 10η [η denotes the Kolmogorov length $\eta=(\nu^3/\epsilon)^{1/4}$], a value which conforms to the findings of Ref. [12], the expectation value of the strain-parameter a is given as $a=\frac{1}{25}(\frac{\epsilon}{\nu})^{1/2}=\frac{1}{25}\frac{1}{\tau_\eta}$. Note that by this relation a connection between typical scales of the velocity field of the Burgers vortex and physical properties of the flow is established. Moreover, the straining is thereby related to the Kolmogorov time scale τ_η . A typical value of the circulation has to be determined in the same manner. We start with the well-known relation for the Reynolds number on the dissipative scale of the flow $Re_{diss}=\frac{\eta u_\eta}{\nu}\approx 1$. The typical velocity u_η can be approximated by the velocity field of the Burgers vortex on the scale η ,

$$u_\eta = u_\varphi(\eta) = \frac{1}{2\pi\eta}(1 - e^{-a\eta^2/4\nu}) \approx \frac{\Gamma a \eta}{8\pi\nu}. \quad (22)$$

This yields $\Gamma = \frac{8\pi}{a}(\epsilon\nu)^{1/2} = 200\pi\nu$, which determines a typical value for Γ . The distribution of Γ is assumed to be log-normal according to the observations of Ref. [10]. The choice of the typical value for Γ determines the Reynolds number in our model, $R_\Gamma = \frac{\Gamma}{\nu} = 200\pi$.

We note in passing that we have checked the results for different distribution functions of a and Γ . Our results are quite robust against the particular choice of the function. However, the choice of the log-normal distribution is physically motivated and leads to the best results. Now all properties of the model are specified and we are able to perform a numerical simulation.

Figure 1 shows a typical particle path. It reveals that the path is composed of a sequence of vortex trapping events, each of them characterized by a different circulation Γ . Events with a strong circulation can easily be identified by their rapid oscillation of the x component, whereas events with small values of Γ do not cause the particle to perform a complete oscillation. The events with a low value of Γ therefore account to periods, where the particle is not trapped in a strong vortex, whereas periods with a high value of Γ account for the latter.

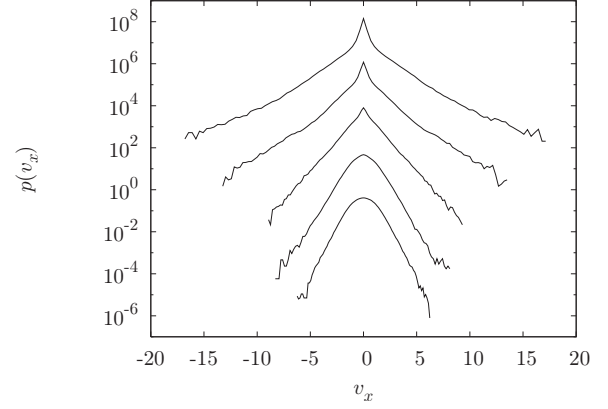


FIG. 2. Logarithmic plot of the PDFs of the velocity increments. The transition from fat-tail PDFs to more Gaussian ones is evident. From upper to lower curve: time lags $\tau\epsilon\{(0.98, 1.97, 3.94, 7.87, 15.74)\tau_\eta\}$. The PDFs are normalized to $\sigma=1$ and shifted vertically.

The switching between vortices renders the velocities discontinuously. This discontinuity can be eliminated using a low-pass filter. A low-pass filtering of the velocity signal with a Butterworth filter was performed and the resulting statistics was compared to the unfiltered one. It was ensured that this effect has no significant impact on the statistics.

One of the central results of our model is depicted in Fig. 2 which shows the probability distribution of the velocity increments. These exhibit a similar behavior to the velocity increment distributions observed in experiments or direct numerical simulation; we will compare the results to experimentally obtained ones in Sec. IV. For small time lags τ the PDFs show fat tails, whereas a nearly Gaussian shape arises for large time lags τ . The physical explanation is straightforward; while the functional form for the short time lags is almost solely determined by a single vortex, the transition to a more Gaussian functional form results from an increasing statistical independence as more vortices are involved. The transition from one functional form to the other is characterized in more detail by the kurtosis, which is shown in Fig. 3. This figure clearly indicates the transition from a highly in-

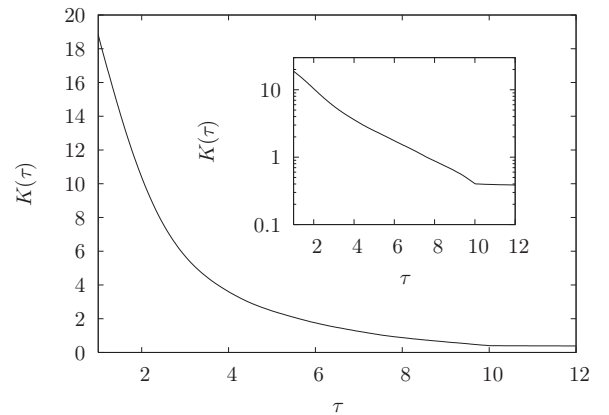


FIG. 3. Kurtosis $K(\tau) = \frac{\langle v_x(\tau)^4 \rangle}{\langle v_x(\tau)^2 \rangle^2} - 3$ for $p(v_x)$, τ is given in multiples of τ_η . The inset shows a log-plot of $K(\tau)$. The kurtosis shows an almost exponential decay.

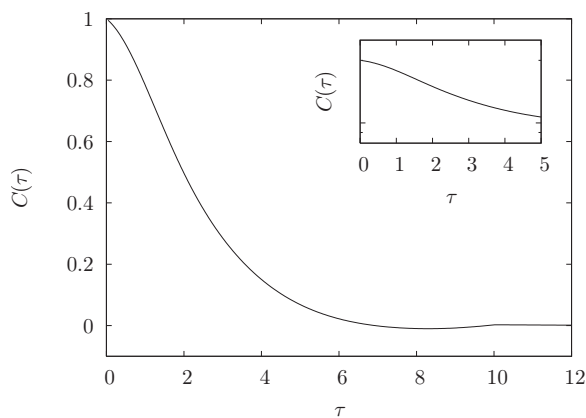


FIG. 4. Autocorrelation function of the velocity u_x , τ is given in multiples of τ_η . The inset shows a log-plot of the vertically shifted autocorrelation function.

mittent distribution to a more Gaussian one.

Since the subsequent trapping events are statistically independent, the velocity signal has to decorrelate as a function of the time delay τ . We define the velocity autocorrelation function as

$$C(\tau) = \frac{\langle u_x(t + \tau)u_x(t) \rangle}{\langle u_x^2 \rangle}. \quad (23)$$

Figure 4 confirms this view; the correlation decreases almost exponentially. The slight anticorrelations might tribute to the events with strong circulations, where the particle is able to perform one or more complete oscillation. The velocity signal decorrelates after a time of $\tau=10\tau_\eta$ which is the lifetime of the vortices. A second effect which cannot be clearly separated is dephasing. As the particle moves inward it constantly alters its angular velocity. It can be shown easily that this also causes the autocorrelation function to decrease.

IV. DISCUSSION

At this point, we would like to comment on similarities and differences with experimental results. We take the results presented in Refs. [6] and [7] as a reference. Comparing the velocity increment distributions, our model reproduces the transition from a Gaussian for large time lags to a fat-tail PDF for small time lags. Our model produces extreme events as large as twenty standard deviations from the mean value in good agreement with the experimental results presented in Ref. [6]. Although the shape of the PDFs is comparable to the experimental results, slight deviations are visible. In our model, the PDFs exhibit a more pronounced cusp for vanishing velocity increments; the experimentally obtained PDFs turn out to be a bit smoother in this region. This difference may result from the fact that our particle path is truly deterministic within one vortex and randomness is only added by choosing a subsequent vortex with random properties. In a real turbulent flow, however, not only the velocity field of the nearest vortex filament is relevant, but also velocity fluctuations from the overall field. One probably could account for this by adding a random noise to the trajectories, which is

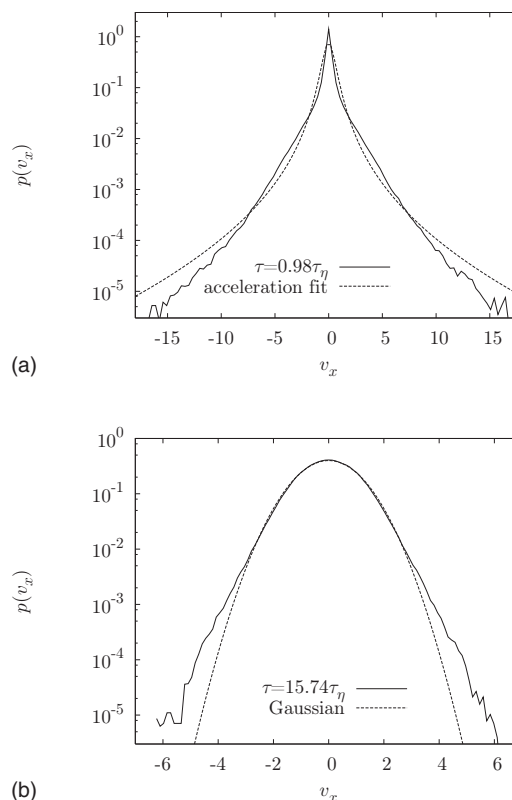


FIG. 5. (a) Comparison of the velocity increment PDF for $\tau=0.98\tau_\eta$ with the fit of the acceleration presented in Ref. [7]. (b) Comparison of the velocity increment PDF for $\tau=15.74\tau_\eta$ with a Gaussian.

small compared to the amplitude of the velocity signal. This would smooth out the cusp for vanishing velocity increments, but leave the overall characteristics of the PDF more or less untouched. As our objective is not to deliver a perfect fit to experimental or numerical results, but to construct a most simple model reproducing the observed transition of the velocity increment PDFs, we did not incorporate this effect. As the PDF of the velocity increments on the smallest time scales can be compared to the acceleration PDF, Fig. 5 shows a comparison with a stretched exponential fit presented in Ref. [7]. The PDF of our velocity increments turns out to be not as stretched as the one presented in Ref. [7], however, the overall functional form compares reasonably well. For large time lags, the experimental results suggest a relaxation of the velocity increment PDF to a Gaussian. This comparison also is made in Fig. 5. Although there is a reasonably good agreement for small values of v_x , slight deviations become apparent for larger values. This is related to the fact that the stationary distribution of the velocity in the model is not perfectly Gaussian.

A second limitation of the present model concerns the proper modeling of inertial range properties. Comparing the evolution of the kurtosis of the velocity increment distribution obtained from our model with the experimental results in Refs. [5,6], deviations are apparent as the experimental results do not suggest an exponential decay of the kurtosis. This leads to the conclusion that our model does not account for inertial range physics properly. This is not very surpris-

ing, as only the small scale vortices of the turbulent flow are modeled and influences of the larger scales are neglected. However, in the classical picture of the energy cascade, the interaction of all scales contributes to the inertial range, and hence this cannot be accounted for in the present model. More statistically speaking, the present model cannot account for the complex multipoint properties (in a spatial and temporal sense) which are necessary to describe a turbulent flow completely. Our model reproduces the characteristic decay of the autocorrelation functions quite well, even the anticorrelations observed in experiments. In our model the anticorrelations can be interpreted physically.

V. SUMMARY

To conclude, we modeled the particle evolution in a turbulent flow as a temporal sequence of Burgers vortices. First calculating the path of a Lagrangian particle in an isolated vortex, we derived the functional form of the velocity increment PDF. This calculation revealed the dependence of the structure of the velocity increment distribution on the physical parameters of the vortex filament such as strain, circulation, etc. A numerical implementation of the model was used to investigate some typical Lagrangian properties of turbulent flows. To this end we had to take into account a set of physical parameters for which we incorporated numerical or experimental data whenever possible. As a central Lagrangian observable we focused on the velocity increment distribution finding that our model qualitatively resembles the velocity increment statistics in real turbulent flows. A further investigation of the corresponding kurtosis clearly indicated intermittent characteristics. As is apparent from the exponential decay of the kurtosis, our model lacks correct inertial range properties. This is consistent with the setup of our

model, as by the present approach we model intermittency in the dissipative scales of turbulence.

Additionally, the velocity autocorrelation function was investigated, revealing an almost exponential decrease and even slight anticorrelations. Consistent with the framework of the model is the fact that the autocorrelation function vanishes for times longer than the lifetime of a single vortex.

The physical interpretation of the observed results is quite straightforward. A Lagrangian particle encountering subsequent trapping events shows fat-tail velocity increment distributions accountable to a proper statistical ensemble of Burgers vortices. For small time lags the fat-tail statistics originate from the superposition of the dynamics in a single vortex. The transition to a Gaussian behavior is caused by an increasing mixing of two subsequent vortex trapping events which are statistically independent. This observation is supported by the functional form of the velocity autocorrelation function.

This model qualitatively reproduces some of the typical Lagrangian statistics. It is important to note that this is achieved by a temporal sequence of exact solutions of the Navier-Stokes equation. This is in contrast to many other models, which apply stochastic equations for the particle evolution. Modeling the velocity field by genuine solutions of the Navier-Stokes equations, our model exhibits realistic looking trajectories. Due to the simplicity of the model, our results can be interpreted in physical terms and shed light on the connection between dynamical aspects in turbulent flows and corresponding statistical properties.

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